IDEA: Generalize Green's theorem to surfaces which are not flat...

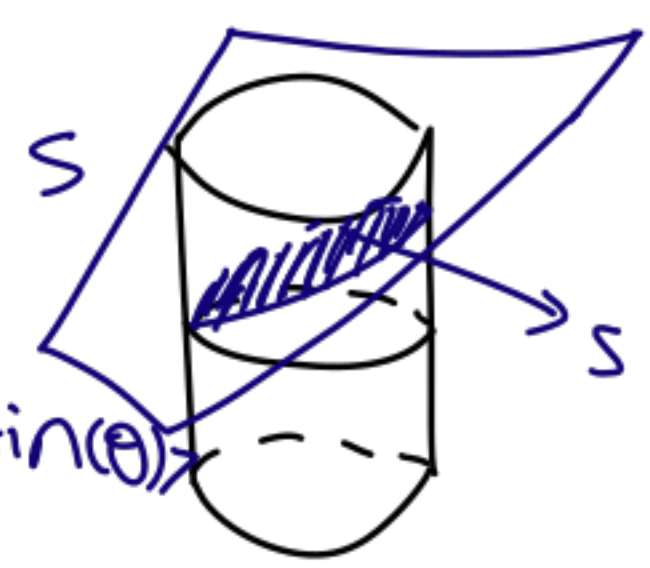
Prop (Stokes's Theorem):

Suppose S is a piecewise smooth surface with piecewise-smooth boundary curve. which is closed and has only one component. If a vector field with continuous partial derivatives on S,

then $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F}\, d\vec{r}$

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C the curve of intersections of plane $y+z=2$ and cylinder $x^2+y^2=1$

Sol: We need $C = \partial S$ for some surface S



A good choice:

$\vec{s}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 2 - r\sin(\theta) \rangle$

on $(r,\theta) \in [0,1] \times [0, 2\pi]$

By Stokes's theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_C \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \iint_D \operatorname{curl}(\vec{F})(\vec{s}(r,\theta)) \cdot (\vec{s}_u \times \vec{s}_\theta)\, dA$$

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y \rangle$$

$$\operatorname{curl}(\vec{F})(S(r,\theta)) = \langle 0, 0, 1+2r\sin(\theta)\rangle$$

$$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -\sin(\theta)\rangle$$

$$\vec{S}_\theta = \langle -r\sin(\theta), r\cos(\theta), -r\cos(\theta)\rangle$$

$$\therefore \vec{S}_r \times \vec{S}_\theta = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) & -r\cos(\theta) \end{bmatrix}$$

$$= r\langle 0, 1, 1\rangle$$ has correct orientation for counterclockwise from above.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_D \langle 0, 0, 1+2r\sin(\theta)\rangle \cdot r\langle 0,1,1\rangle \, dA$$

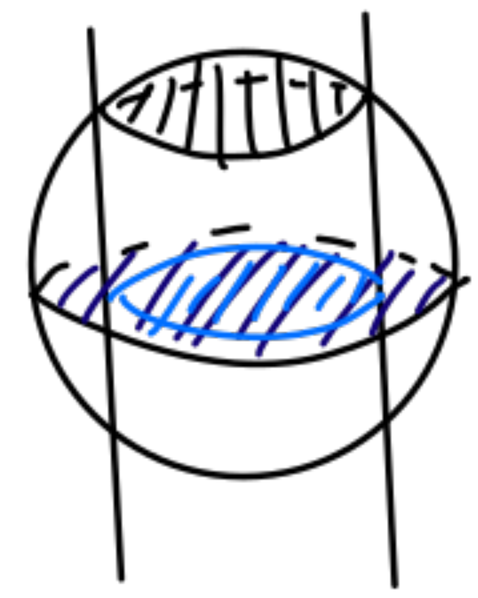
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r(1+2r\sin(\theta))\, d\theta\, dr$$

$$= \pi$$

<u>Exercise</u>: Directly compute the line integrals...

NB: Often curl($\vec{F}$) is simpler than $\vec{F}$.

EX: Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ and S the part of sphere $x^2+y^2+z^2=4$ inside cylinder $x^2+y^2=1$ and above xy-plane.



Sol 1: (w/o Stokes's Theorem)

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & xy \end{vmatrix} = (x-y)(1,1,0)$$

parameterize S (in cylindrical coordinates):

$$\vec{s}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{4-r^2} \rangle \text{ on } (r,\theta) \in [0,1] \times [0,2\pi]$$

$$z^2 = 4 - r^2$$

$$\vec{s}_r = \langle \cos(\theta), \sin(\theta), -\tfrac{1}{2}(4-r^2)^{-1/2} \cdot 2r \rangle$$
$$= \langle \cos(\theta), \sin(\theta), -r(4-r^2)^{-1/2} \rangle$$

$$\vec{s}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$$

$$\vec{s}_r \times \vec{s}_\theta = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -r(4-r^2)^{-1/2} \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{bmatrix}$$
$$= \langle r^2(4-r^2)^{-1/2}\cos(\theta), r^2(4-r^2)^{-1/2}\sin(\theta), r \rangle$$

$$\text{curl}(\vec{F})(\vec{s}(r,\theta)) = (r\cos(\theta) - r\sin(\theta))(1,1,0)$$

$$\therefore \iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \iint_D \operatorname{curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)\, dA$$

$$= \iint_D (r\cos(\theta) - r\sin(\theta))\langle 1,1,0\rangle \cdot \langle r^2(4-r^2)^{-1/2}\cos(\theta), r^2(4-r^2)^{-1/2}\sin(\theta), r\rangle\, dA$$

$$= 0$$

Sol 2: w/ Stoke's theorem

$$\iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

Parameterize ∂S via $\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta), \sqrt{3}\rangle$

$$\vec{F}(\vec{r}(\theta)) = \langle \sqrt{3}\cos(\theta), \sqrt{3}\sin(\theta), \sin(\theta)\cos(\theta)\rangle$$

$$\vec{r}\,'(\theta) = \langle -\sin(\theta), \cos(\theta), 0\rangle$$

$$\therefore \iint_S \operatorname{curl}(\vec{F})\, d\vec{S} = \int_{\theta=0}^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, d\theta$$

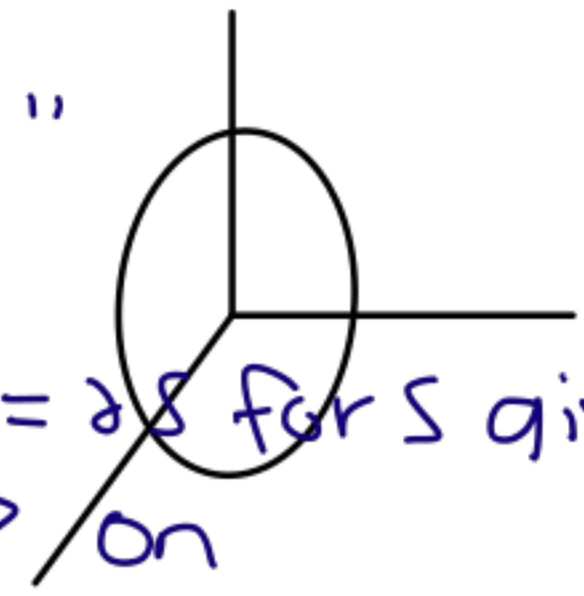
$$= \int_{\theta=0}^{2\pi} \left(-\sqrt{3}\cos(\theta), \sqrt{3}\sin(\theta), \sin(\theta)\cos(\theta)\right) \cdot \langle -\sin(\theta), \cos(\theta), 0\rangle = 0$$

NB: the "Stokes Equation" also implies

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_T \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

whenever $\partial S = \partial T$...

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle xy, yz, zx \rangle$
and C the boundary of the part of
$z = 1 - x^2 - y^2$ in the first octant.



Sol: Note that C has "three process"
(i.e. it is piecewise-defined)
Let's try Stokes's Theorem: $C = \partial S$ for S given
by $\vec{s}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 1-r^2 \rangle$ on
$(r,\theta) \in [0,1] \times [0, \frac{\pi}{2}]$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & zx \end{bmatrix}$$

$$= \langle y, z, x \rangle$$

$$\text{curl}(\vec{F})(\vec{s}(r,\theta)) = -\langle \sin(\theta), 1-r^2, \cos(\theta) \rangle$$

$$\vec{s}_r = \langle \cos(\theta), \sin(\theta), -2r \rangle, \quad \vec{s}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$$

$$\vec{s}_r \times \vec{s}_\theta = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{bmatrix} = r\langle 2r\cos(\theta), 2r\sin(\theta)$$

$$\therefore \text{curl}(\vec{F})(\vec{s}(r,\theta)) \cdot (\vec{s}_r \times \vec{s}_\theta)$$
$$= -r(2r^2\sin(\theta)\cos(\theta) + 2(1-r^2)r\sin(\theta) + r\cos(\theta))$$
$$= -r^2(r\sin(2\theta) + 2(1-r^2)\sin(\theta) + \cos(\theta))$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\, d\vec{s}$$
$$= \iint_D \text{curl}(\vec{F})(\vec{s}(r,\theta))(\vec{s}_r \times \vec{s}_\theta)\, dA$$
$$= \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} -r^2(r\sin(2\theta) + 2(1-r^2)\sin(\theta) + \cos(\theta))\, d\theta\, dr$$
$$= \frac{2}{5} - \frac{1}{4} - 1$$